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Abstract. Analytical expressions are given for the eigenvalues and eigenvectors of a Hamiltonian with $su_q(2)$ dynamical symmetry. The relevance of such an operator in quantum optics is discussed. As an application, the ground-state energy in the Dicke model is studied through $su_q(2)$ perturbation theory.

1. Introduction

Many models in quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians of the form (see, e.g., [1–4])

$$H = \begin{pmatrix} 0 & A_l & 0 & \cdots & 0 \\ A_l & 0 & A_{l-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & A_{-l+2} & 0 & A_{-l+1} \\ 0 & \cdots & 0 & A_{-l+1} & 0 \end{pmatrix}. \tag{1.1}$$

The dimension $d = (2l + 1)$ of this matrix is by no means small (for instance, in the Dicke model, $2l$ is the number of atoms considered). Therefore, finding analytical expressions for the corresponding eigenvalues and eigenvectors of H is essential in order to solve the dynamics of the model. It is also important to point out that, in some cases, H can be seen as a perturbation of the J_x generator of an underlying $su(2)$ dynamical symmetry. This fact has been successfully used in order to describe many features of these models [3], and it will be also relevant in what follows.

In this paper we show that the $su_q(2)$ quantum algebra (see, e.g., [5–13]) can be used to define a Hamiltonian of the type (1.1) in a natural way. Such a Hamiltonian is introduced as a simple function of the $su_q(2)$ generators having as non-deformed limit the J_x generator of the $su(2)$ algebra. By considering the well known representation theory of $su_q(2)$ (which is revisited in section 2), the Hamiltonian is defined and its eigenvalues and eigenvectors are found (section 3). The spectrum obtained is essentially anharmonic; thus we have a new exactly solvable nonlinear quantum model with $su_q(2)$ dynamical symmetry.

In section 4, relevant Clebsch–Gordan coefficients for both the bare and the dressed basis are considered. Finally, in section 5 we present an application of the previous analytical results

to finding the energy of the ground state of the Dicke model by making use of a perturbation theory around the $su_q(2)$ Hamiltonian. This preliminary study shows that the q-deformed J_x operator here introduced is physically meaningful in the context of the quantum optical Hamiltonians mentioned above (compare with the results given in [2–4], for example), and the explicit dynamical features of the $su_q(2)$ Dicke model will be fully developed in a forthcoming paper. In this paper, we shall provide the basic algebraic properties that are needed in order to solve it explicitly. It is interesting to stress that some of these properties show new (to our knowledge) features of the $su_q(2)$ algebra, all of them related to the deformed $J_x = (J_+ + J_-)/2$ generator.

2. The $su_q(2)$ algebra

Let the operators J_z, $J_\pm$ generate the quantum algebra $su_q(2)$ with the coproduct [5–13]

$$\Delta(J_\pm) = J_\pm \otimes q^{-J_z} + q^{J_z} \otimes J_\pm \tag{2.1}$$

$$\Delta(J_z) = J_z \otimes 1 + 1 \otimes J_z. \tag{2.2}$$

The deformed commutation rules consistent with the previous map are given in the form

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2 J_z] \tag{2.3}$$

where $[x] := (q^x - q^{-x})/(q - q^{-1})$ and $q = e^{z/2}$. We shall assume that q is not a root of unity, and we shall recover the 'classical' results when $q \to 1$.

Let us introduce, firstly, the 'bare' basis of the eigenvectors of J_z:

$$2J_z|l, m\rangle = 2m|l, m\rangle. \tag{2.4}$$

In this basis, the $(2 l + 1)$-dimensional irreducible representation of $su_q(2)$ is given by (2.4) and

$$J_\pm|l, m\rangle = \sqrt{[l \mp m][l \pm m + 1]}|l, m+1\rangle. \tag{2.5}$$

Throughout the paper we will use the following (standard) notation for the q-numbers:

$$[n] = q^{-n+1} + q^{-n+3} + q^{-n+5} \cdots + q^{n-1} = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [0] = 0, \quad [-n] = -[n]. \tag{2.6}$$

We shall also introduce the symbols

$$[n, m] := q^{-(n-1)m} + q^{-(n-3)m} + q^{-(n-5)m} + \cdots + q^{(n-1)m} = [nm]/[m]. \tag{2.7}$$

Thus, $\lim_{q \to 1}[n, k] = n$, and

$$[n, 1] = [n], \qquad [k, 0] = k, \qquad [2, k] = q^k + q^{-k} = [2k]/[k]. \tag{2.8}$$

3. The Hamiltonian

Let us consider the following $su_q(2)$ operator as a Hamiltonian:

$$H = q^{J_z/2} (J_+ + J_-) q^{J_z/2}. \tag{3.1}$$

We stress that the $q \to 1$ limit of H is just $2 J_x$, but H is *not* the standard way to introduce the deformed J_x operator in $su_q(2)$. On the other hand, one can easily check that, from the definition (3.1) and the maps (2.1)-(2.2), the coproduct of H can be deduced (note that Δ is an algebra homomorphism: $\Delta(X Y) = \Delta(X) \Delta(Y)$):

$$\Delta(H) = H \otimes 1 + q^{2J_z} \otimes H. \tag{3.2}$$

Such a form of the coproduct is known (see, e.g., [13]). In the $(2l + 1)$-dimensional representation D_l, H thus takes the form (1.1) with

$$A_m = q^{m-1/2}\sqrt{[l+m][l-m+1]}. \tag{3.3}$$

In particular, when $l = 1$, we have

$$D_1(H) = \begin{pmatrix} 0 & q^{1/2}\sqrt{[2]} & 0 \\ q^{1/2}\sqrt{[2]} & 0 & q^{-1/2}\sqrt{[2]} \\ 0 & q^{-1/2}\sqrt{[2]} & 0 \end{pmatrix}. \tag{3.4}$$

A straightforward computation shows that the spectrum of this operator is $[2]$, 0, $-[2]$. The corresponding normalized eigenvectors are

$$|\underline{1,\pm1}\rangle = \frac{1}{\sqrt{2[2]}}\begin{pmatrix} q^{1/2} \\ \pm\sqrt{[2]} \\ q^{-1/2} \end{pmatrix} \qquad |\underline{1,0}\rangle = \frac{1}{[2]}\begin{pmatrix} q^{-1/2}\sqrt{[2]} \\ 0 \\ -q^{1/2}\sqrt{[2]} \end{pmatrix}. \tag{3.5}$$

Due to the richness of the structures underlying quantum deformations, this remarkable anharmonic deformation of the J_x Hamiltonian can be explicitly solved for arbitrary l as follows.

3.1. Spectrum and eigenvectors

It can be proved (by induction and using the coproduct to construct higher-dimensional representations from the three-dimensional one considered before) that the spectrum of this operator for a given l is just $[2m]$, with $m = -l, \ldots, l$.

Moreover, the eigenvector $|\underline{l,l}\rangle'$ corresponding to the highest eigenvalue $[2l]$ is given by the following components in the bare basis $|l, m\rangle$:

$$\alpha_{ml} \equiv \langle l, m|\underline{l,l}\rangle' = q^{m(l-1/2)}\sqrt{\frac{[2l]!}{[l+m]![l-m]!}}. \tag{3.6}$$

Here, the prime means that a special normalization is accepted, temporarily, where $\alpha_{ll} = q^{l(l-1/2)} = 1/\alpha_{-l-l}$. The next eigenvector $|\underline{l,l-1}\rangle'$ with the eigenvalue $[2l-2]$ is given by

$$\alpha_{m,l-1} = \langle l, m|\underline{l,l-1}\rangle' = \alpha_{ml}\, q^{-2m+1}\sqrt{[2l]}\left(1 - q^{l+m-1}[2]\frac{[l-m]}{[2l]}\right). \tag{3.7}$$

Finally, an arbitrary eigenvector with eigenvalue $[2m]$ can be deduced, namely

$$\alpha_{mn} = \langle l, m|\underline{l,n}\rangle' = \alpha_{ml}\, q^{(l-n)(l-n-2m)}\sqrt{\frac{[2l]!}{[l-n]![l+n]!}}$$

$$\times \sum_{j=0}^{l-n}\left((-1)^j q^{-j(l-m-2n)+j(j+1)/2}\frac{[2(l-n)]!!}{[j]![2(l-n-j)]!!}\right.$$

$$\left.\times \frac{[l-m][l-m-1]\cdots[l-m-j+1]}{[2l][2l-1]\cdots[2l-j+1]}\right) \tag{3.8}$$

where $[2n]!! := [2n]\cdot[2n-2]\cdots[2]$.

3.2. Normalization

These eigenvectors can be easily normalized in terms of q-numbers. For the ground and top states we have

$$|l, l\rangle = \mathcal{N}_{l,l}^{-1} |l, l\rangle' \qquad \text{where} \qquad \mathcal{N}_{l,l}^2 = \mathcal{N}_{l,-l}^2 = {}'\langle l, l|l, l\rangle' = q^{-l(l-1/2)} \sum_{k=0}^{2l} \frac{q^{k(2l-1)}[2l]!}{[k]![2l-k]!}. \tag{3.9}$$

A simple calculation gives

$$\mathcal{N}_{l,l}^2 = \prod_{k=0}^{2l-1}(q^k + q^{-k}) = 2\,[2]\,(q^2 + q^{-2})\cdots(q^{2l-1}+q^{-2l+1}) = \prod_{k=0}^{2l-1}[2, k]. \tag{3.10}$$

The excited states can be normalized as follows:

$$\mathcal{N}_{l,n}^2 = {}'\langle l, n|l, n\rangle' = \frac{1}{q^{2n} + q^{-2n}} \prod_{k=-l+n}^{l+n}\left(q^k + q^{-k}\right) = \frac{1}{[2, 2n]} \prod_{k=-l+n}^{l+n}[2, k]. \tag{3.11}$$

For instance

$$\mathcal{N}_{1/2,1/2}^2 = \mathcal{N}_{1/2,-1/2}^2 = 2, \quad \mathcal{N}_{1,1}^2 = \mathcal{N}_{1,-1}^2 = 2\,[2], \quad \mathcal{N}_{1,0}^2 = [2]^2, \quad \dots\,. \tag{3.12}$$

Note that all of the coefficients $\mathcal{N}_{l,n}^2$ go to 2^{2l} in the limit $q \to 1$. Finally, if we denote the entries of the normalized eigenvectors as

$$\mathcal{A}_{mn}^l = \alpha_{mn}^l/\mathcal{N}_{ln} \tag{3.13}$$

the following relations hold:

$$\mathcal{A}_{mn}^l(q) = (-1)^{l-n}\mathcal{A}_{-m,n}^l(1/q) \qquad \mathcal{A}_{mn}^l(q) = (-1)^{l-m}\mathcal{A}_{m,-n}^l(q). \tag{3.14}$$

They generalize the known symmetry of the non-deformed $su(2)$ case ($q = 1$) by involving the transformation $q \to q^{-1}$.

3.3. Orthogonality relations

For physical applications (like the calculation of mean values) is desirable to further develop the previous 'q-arithmetics'. With this goal in mind, let us note that the quantities $|\alpha_{ml}|^2$ from (3.6) play the role of the binomial coefficients. In particular, the moments of this deformed binomial distribution, i.e. the mean values

$$\langle Y_k \rangle_{2l} \equiv \sum_{k=0}^{2l} |\alpha_{k-l,l}|^2\, Y_k \tag{3.15}$$

can be calculated. With this definition, the equation (3.9) is then rewritten in the form

$$\langle 1 \rangle_{2l} \equiv \prod_{s=0}^{2l-1}(q^s + q^{-s}). \tag{3.16}$$

Moreover, one has the following deformed formulae ($0 \leqslant j, \ M \leqslant 2l$) for the binomial distribution:

$$\langle q^{-2jk}\rangle_{2l} = q^{-2jl} \prod_{s=-j}^{2l-j-1}(q^s + q^{-s}) \tag{3.17}$$

$$\langle [k][k-1]\cdots[k-M+1]\,q^{-k(2j+M)}\rangle_{2l} = q^{-2Mj-M(M+1)/2}\,\langle q^{-2jk}\rangle_{2l-M} \tag{3.18}$$

$$= [2l][2l-1]\cdots[2l-M+1]\,q^{-j(2l+M)-M(M+1)/2} \prod_{s=-j}^{2l-M-1-j}(q^s + q^{-s}). \tag{3.19}$$

These formulae are equivalent to the fact that the dressed vectors form an orthonormal basis.

4. Clebsch–Gordan coefficients for the dressed basis

We can decompose the tensor product of the dressed vectors (3.8), (3.13) into the irreducible parts

$$|\underline{l, m}\rangle \otimes |\underline{1, i}\rangle = \sum_{j=l-1}^{j=l+1} \underline{C}^j_{l1,mi} |\overline{j, m+i}\rangle_{12} \tag{4.1}$$

where $\underline{C}^j_{l1,mi}$ are the Clebsch–Gordan (CG) coefficients for the dressed basis. However, the vector $|\underline{l, m}\rangle \otimes |\underline{1, i}\rangle$ on the left-hand side and vectors $|\overline{j, m+i}\rangle_{12}$ on the right-hand side are eigenvectors of $\underline{J}_x$ in the corresponding representations only for $i = 0$. Thus, only the coefficients $\underline{C}^j_{l1,m0}$ are of interest. For $l = 1$, they are given as

$$\underline{C}^2_{11,10} = \sqrt{\frac{q^3 + q^{-3}}{[4]}} \qquad \underline{C}^1_{11,10} = -\sqrt{\frac{1}{q^2 + q^{-2}}} \qquad \underline{C}^0_{11,10} = 0 \tag{4.2}$$

$$\underline{C}^2_{11,00} = \sqrt{\frac{[4]}{[2][3]}} \qquad \underline{C}^1_{11,00} = 0 \qquad \underline{C}^0_{11,00} = -\sqrt{\frac{1}{[3]}} \tag{4.3}$$

$$\underline{C}^2_{11,-10} = \underline{C}^2_{11,10} \qquad \underline{C}^1_{11,-10} = -\underline{C}^1_{11,10} \qquad \underline{C}^0_{11,-10} = 0. \tag{4.4}$$

They thus differ from the coefficients in the bare basis [9] (though have the same limit $q \to 1$).

By rewriting the definition

$$\underline{C}^j_{l1;\,p,0} = {}_{12}\langle \underline{j, p} | \underline{l, p}\rangle_1 |\underline{1, 0}\rangle_2 \tag{4.5}$$

in terms of components of the dressed vectors, and afterwards replacing explicitly the components of $|\underline{1, 0}\rangle_2$, we get the relation

$$\underline{C}^j_{l1;p0} = \frac{1}{\sqrt{[2]}} \sum_{k=-j}^{j} \mathcal{A}^j_{kp} \left(q^{-1/2} C^j_{l1;k-1,1} \mathcal{A}^l_{k-1,p} - q^{1/2} C^j_{l1;k+1,-1} \mathcal{A}^l_{k+1,p} \right). \tag{4.6}$$

Here, $\mathcal{A}^l_{mn} = 0$ if $|m| > L$ and

$$C^j_{l1;m_1,m_2} = {}_{12}\langle j, m_1 + m_2 | l, m_1\rangle_1 |1, m_2\rangle_2 \tag{4.7}$$

are the CG coefficients in the bare basis. Another useful formula for the Clebsch–Gordan coefficients in the dressed basis can be obtained as follows. Starting with the expansion of the tensor product

$$|\underline{l, p}\rangle_1 |\underline{1, 0}\rangle_2 = \sum_j \underline{C}^j_{l1;\,p0} |\underline{j, p}\rangle_{12} \tag{4.8}$$

we can rewrite the dressed vectors in the bare basis, $|\underline{j, p}\rangle_{12} = \sum_{m=-j}^{j} \mathcal{A}^j_{mp} |j, m\rangle_{12}$, and the vectors $|j, m\rangle_{12}$ in terms of the tensor product, using bare CG coefficients:

$$|j, m\rangle_{12} = \sum_{m_2=-1}^{1} C^j_{l1;\,m-m_2,m_2} |l, m - m_2\rangle_1 |1, m_2\rangle_2. \tag{4.9}$$

We have

$$\mathcal{A}^l_{m_1 p} \mathcal{A}^1_{m_2 0} = \sum_{j=l-1}^{l+1} \underline{C}^j_{l1;\,p0} \mathcal{A}^j_{m_1+m_2,p} C^j_{l1;\,m_1 m_2}. \tag{4.10}$$

Multiplying by $C^k_{l_1l_2;\,m_1,m_2}$, making a summation under the condition $m_1 + m_2 = m$, using the orthogonality of the Clebsch–Gordan coefficients

$$\sum_{m_2} C^k_{l_1l_2;\,m-m_2,m_2} C^j_{l_1l_2;\,m-m_2,m_2} = \delta_{jk} \tag{4.11}$$

and, finally, replacing the explicit form of $\mathcal{A}^1_{m0}$, we have the connection between the coefficients in the bare and dressed basis:

$$\underline{C}^j_{l1;\,p0}\mathcal{A}^j_{mp} = \frac{1}{\sqrt{[2]}}\left\{ -\mathcal{A}^l_{m+1,p}\, q^{1/2}\, C^j_{l1;\,m+1,-1} + \mathcal{A}^l_{m-1,p}\, q^{-1/2}\, C^j_{l1;\,m-1,1} \right\}$$

$$-j \leqslant m \leqslant j \quad -l \leqslant p \leqslant l. \tag{4.12}$$

For instance, using these formulae for $m = j$, we can express the 'dressed' coefficients in terms of 'bare' ones.

Finding the components of the first three rows of the matrix $\mathcal{A}^l_{mn}$

$$\mathcal{A}^l_{ln} = \frac{q^{n^2-l/2}}{\mathcal{N}_{ln}} \sqrt{\frac{[2l]!}{[l-n]![l+n]!}}$$

$$\mathcal{A}^l_{l-1,n} = \frac{q^{n^2-(3l-1)/2}}{\mathcal{N}_{ln}} \sqrt{\frac{[2l]!}{[l-n]![l+n]!}}\, \frac{[2n]}{\sqrt{[2l]}} \tag{4.13}$$

$$\mathcal{A}^l_{l-2,n} = \frac{q^{n^2-l/2+1}}{\mathcal{N}_{ln}} \sqrt{\frac{[2l]!}{[l-n]![l+n]!}}\, \frac{q^{1-2l}[2n]^2 - [2l]}{\sqrt{[2l][2l][2l-1]}}$$

and using equation (4.12), we arrive at the explicit expressions

$$\underline{C}^{l+1}_{l1;\,p0} = \sqrt{\frac{[l+1+p][l+1-p][2,l+1+p][2,l+1-p]}{[2][2l+2][2l+1]}}$$

$$\underline{C}^l_{l1;\,p0} = -\frac{[2p]}{\sqrt{[2l+2][2l]}} \tag{4.14}$$

$$\underline{C}^{l-1}_{l1;\,p0} = -\sqrt{\frac{[l+p][l-p][2,l+p][2,l-p]}{[2][2l][2l+1]}}.$$

5. Application to the Dicke model

In order to demonstrate how our approach works we will apply it to the Dicke model, which describes the interaction of a system of $N = 2l$ two-level atoms with the quantum radiation field in an ideal cavity. (This model is mathematically equivalent to the three-photon Hamiltonian describing three-wave mixing.) The Hamiltonian can be written in the matrix form (1.1) with the matrix elements

$$B_m = \sqrt{(l+m)(l+1-m)(s+1-m)} \tag{5.1}$$

where $s \geqslant l$ is a parameter ($2s$ is the excitation number which is conserved in this model [3]). We restrict ourselves to the case of the highest nonlinearity, $s = l$. (In the language of three-wave mixing processes this corresponds to the second harmonic generation.) We are particularly interested in the limit of large l, which corresponds to high photon numbers.

We will take the Hamiltonian

$$H_0 = \Omega H \tag{5.2}$$

with H (3.1) belonging to $su_q(2)$ as a zeroth-order Hamiltonian, and we will find the Dicke spectrum by using the perturbation theory. In the present paper we restrict ourselves to the energy of the ground state for the sake of simplicity. The values of q and Ω are to be chosen. The simplest way to fit them is to provide the coincidence of the points where the matrix elements B_m of the three-wave Hamiltonian and the matrix elements

$$A_m = \Omega\, q^{m-1/2}\sqrt{[l-m][l+m+1]} = \Omega\, \langle l, m+1|H|l, m \rangle \tag{5.3}$$

of the Hamiltonian H_0 take their maximum values. It gives (for $s = l$)

$$\alpha = N \log q = \frac{3}{2}\log\frac{\sqrt{5}-1}{2} \approx -0.7218 \tag{5.4}$$

and the maxima of $B(m)$ and $A(m)$ occur in the point $m_0 = -(l-1)/3$. We chose the coefficient Ω to make equal the values of A_m and B_m in their maxima. This gives

$$\Omega = \frac{4(N+1)^{3/2}}{\sqrt{27}[N+1]}. \tag{5.5}$$

At the next step, we find the approximation for the three-wave Hamiltonian in the form

$$B_m \approx \Omega\, A_m\, \phi(m) \qquad \phi(m) = 1 + \phi_1\Delta - \phi_2\Delta^2 + \phi^3\Delta^3 \qquad \Delta = m - m_0. \tag{5.6}$$

We thus restrict the expansion up to the third-order polynomial $\phi(m)$. We can find the coefficients ϕ_j explicitly by comparing the Taylor expansions for the matrix elements $B(m)$ and $A(m)$ around the point m_0. We have

$$B(m) = \frac{2(N+1)^{3/2}}{\sqrt{27}}\left[1 - \frac{27}{8}\left(\frac{\Delta}{N+1}\right)^2 + \frac{27}{8}\left(\frac{\Delta}{N+1}\right)^3 + O(N^{-4})\right] \tag{5.7}$$

and

$$A(m) = \frac{[N+1]}{2}\left[1 - \frac{2\alpha^2}{\tanh^2\alpha}\left(\frac{\Delta}{N+1}\right)^2 - \frac{4\alpha^3}{\tanh^2\alpha}\left(\frac{\Delta}{N+1}\right)^3 + O(N^{-4})\right] \tag{5.8}$$

which determines the polynomial $\phi(\Delta)$:

$$\phi(\Delta) = 1 - \left(\frac{27}{8} - \frac{2\alpha^2}{\tanh^2\alpha}\right)\left(\frac{\Delta}{N+1}\right)^2 + \left(\frac{27}{8} + \frac{4\alpha^3}{\tanh^2\alpha}\right)\left(\frac{\Delta}{N+1}\right)^3 + O(N^{-4}). \tag{5.9}$$

Now we may substitute $\Delta = m - m_0 = J_z + (l-1)/3$ and rewrite (5.6) in matrix form:

$$H_D \approx \Omega\left[J_+\phi(J_z - m_0) + \phi(J_z - m_0)J_-\right] = 2\Omega\{J_x, f(J_z)\}. \tag{5.10}$$

Here $J_{\pm,z}$ are generators of $su_q(2)$ and $\{A, B\} = AB + BA$. The new function $f(J_z)$ is also a polynomial of degree three, whose coefficients can be easily found. Now the ground-state energy is approximately given as

$$\langle \underline{-l}, l|H_D|\underline{-l}, l \rangle \approx -\Omega[2l]\sum_{k=0}^{3} f_k\langle \underline{-l}, l|(J_z)^k|\underline{-l}, l \rangle. \tag{5.11}$$

Therefore, we have reduced the problem to the calculation of the averages of the powers of the operators J_z (the moments) in the eigenstates of the operator H.

Though this problem can be solved for arbitrary eigenstates, here we consider the ground states only. By using the results of the previous sections we can write the generating function for these moments. For arbitrary $p = \exp\mu$ and $q = \exp t$ we find

$$\langle \underline{-l}, l|p^{2J_z}|\underline{-l}, l \rangle = \prod_{k=0}^{N-1}\frac{\cosh(\mu + kt)}{\cosh(kt)}. \tag{5.12}$$

Now, let us introduce the notation

$$S_k = \sum_{j=1}^{N-1} \tanh^k jt \approx \int_1^{N-1} \mathrm{d}x \ \tanh^k xt + \frac{1}{2}\big[\tanh^k t + \tanh^k(N-1)t\big] \tag{5.13}$$

where we have used the Euler–Maclaurin summation formula. Differentiating (5.12) with respect to μ, we find

$$\langle \underline{-l,l}|2J_z|\underline{-l,l}\rangle = S_1 \tag{5.14}$$

$$\langle \underline{-l,l}|(2J_z)^2|\underline{-l,l}\rangle = S_1^2 - S_2 + N\,, \tag{5.15}$$

$$\langle \underline{-l,l}|(2J_z)^3|\underline{-l,l}\rangle = S_1^3 - 3S_2 S_1 + (3N-2)S_1 + 2S_3. \tag{5.16}$$

On the other hand, from (5.13) we have the following expressions for the first three sums:

$$S_1 \approx \frac{N}{\alpha} \log\cosh(\alpha - t) + \frac{\tanh(\alpha - t)}{2} \qquad t = \frac{\alpha}{N} = \log q \tag{5.17}$$

$$S_2 \approx N - 1 - N\frac{\tanh(\alpha - t)}{\alpha} + \frac{\tanh^2(\alpha - t)}{2} \tag{5.18}$$

$$S_3 \approx \frac{N}{\alpha} \log\cosh(\alpha - t) - N\frac{\tanh^2(\alpha - t)}{2\alpha} + \frac{\tanh^3(\alpha - t)}{2}. \tag{5.19}$$

The combination of all these formulae gives the approximation for the energy of the ground state. Comparing with the numerical results, we can say that the accuracy for the energy of the ground state is 1.5% for 100 atoms ($N = 100$) and 0.35% for 400 atoms. Note that it is the Maclaurin summation formula (5.13) that reduces the accuracy, which would otherwise be much higher. However, it gives the correct asymptotic behaviour when $N \to \infty$, which is sufficient for our goals. We may also mention that our method produces much better accuracy than the analogous perturbation theory with common $su(2)$ as a dynamical symmetry algebra [3, 14] orthe variational method with the $su(2)$ coherent states as probe states [15]. The complete description of the spectrum and the dynamical analysis of the Dicke model by means of the present approach will be given elsewhere.

Acknowledgments

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